On groups of homotopy spheres

1. The group Θ_n
2. S-parallelizability
3. bP_{n+1} & Θ_n / bP_{n+1}
4. Computation of bP_{2k+1}

Def: $\Theta_n = \left(\{\text{homotopy } n\text{-spheres}\} / h\text{-cobordism} \, , \# \right)$

The goal of this talk is to extract some information about these groups.

The term "manifold" will refer to "smooth, compact, oriented manifold".

Def: A closed n-mfld M is a homotopy n-sphere if $M \simeq S^n$.

Def: Two closed mflds M_1^n, M_2^n are h-cobordant $(M_1 \sim_h M_2)$ if $\exists W^{n+1}$ mfld with $\partial W = M_1 \sqcup (-M_2)$ and both M_1, M_2 are deformation retracts of W.

$\#$ defined later. Now: why should we care about Θ_n?

Thm (Poicaré conjecture): A htpy n-sphere is homeomorphic to S^n

Thm (h-cobordism thm): For $n \geq 5$, $\underbrace{M_1^n \sim_h M_2^n}_{\text{1-connected}} \iff M_1^n \overset{\text{diffeo.}}{\cong} M_2^n$

↑ orientation preserving.

Hence, we can identify

$$\{\text{homotopy } n\text{-spheres}\} \longleftrightarrow \{\text{topological } n\text{-spheres}\}$$

$$\text{h-cobordism} \cong \text{diffeomorphism}$$

to get $\Theta_n \cong \underbrace{\{\text{top. } n\text{-spheres}\}/\text{diffeom.}}_{\text{smooth structures on } S^n.}$ for $n \geq 5$

Some things we can already infer:

- $\Theta_1 = \Theta_2 = 0$
- $\Theta_3 = 0$
- $\Theta_7 \neq 0$ from two talks ago
- $\Theta_4 \neq 0$ from last week.

easy ($n = 1, 2$); hard! Dominated by geometry ($n = 3$); top-high dimension, diff. low dimension ($n = 4$)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\#\Theta_n$	1	1	1	1	1	1	28	2	8	6	992	1	3	2	16256
# smooth structures on S^n	1	1	1	?											

1. The group Θ_n

Def: M_1, M_2 connected n-mflds. Pick embeddings

$$i_1 : D^n \underset{\text{or. pres.}}{\hookrightarrow} M_1 \quad , \quad i_2 : D^n \underset{\text{or. rev.}}{\hookrightarrow} M_2$$

The connected sum $M_1 \# M_2$ is

$$\left(M_1 \setminus i_1(0)\right) \underset{\substack{i_1(tu) \sim i_2((1-t)u) \\ 0<t<1}}{\cup} \left(M_2 \setminus i_2(0)\right)$$

with orientation compatible with M_1 and M_2.

This can be canonically equipped with a smooth structure.

Lemma 1.1: $M_1 \# M_2$ is well-defined up to $\underset{\text{or. pres.}}{\text{diffeom.}}$

Pf) Lemma (Palais, Cerf): two or. pres. embeddings $i, j : D^n \to M$ are related by $j = \varphi \circ i$ with $\varphi : M \to M$ a diffeom.

Suppose we picked embeddings

$$i_1, j_1 : D^n \underset{\text{or. pres.}}{\hookrightarrow} M_1 \quad , \quad i_2, j_2 : D^n \underset{\text{or. rev.}}{\hookrightarrow} M_2$$

$$j_1 = \varphi_1 \circ i_1 \qquad\qquad j_2 = \varphi_2 \circ i_2$$

$$\left(M_1 \setminus i_1(0)\right) \cup_f \left(M_2 \setminus i_2(0)\right) \longrightarrow \left(M_1 \setminus j_1(0)\right) \cup_g \left(M_2 \setminus j_2(0)\right)$$

$$M_1 \setminus i_1(0) \xrightarrow{\varphi_1} M_1 \setminus j_1(0)$$

$$M_2 \setminus i_2(0) \xrightarrow{\varphi_2} M_2 \setminus j_2(0).$$

$\square$

Lemma 1.2: # satisfies:

(i) It is commutative and associative up to diffeom.

(ii) $M \# S^n \cong M$.

(iii) Σ_1^n, Σ_2^n htpy n-spheres $\Rightarrow$ $\Sigma_1^n \# \Sigma_2^n$ htpy n-sphere.

(iv) $M_1 \sim_h M_1' \Rightarrow M_1 \# M_2 \sim_h M_1' \# M_2$

(v) M^n 1-connected. $M \sim_h S^n \iff M$ bounds a contractible mfld

(vi) If Σ is a htpy n-sphere, then $\Sigma \# (-\Sigma) \sim_h S^n$

$\rightsquigarrow$ $(\Theta_n, \#)$ is a well-defined abelian group.

Pf of (v)

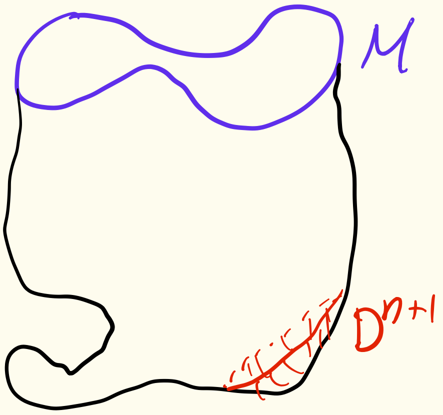
$\Rightarrow$) $M \sim_h S^n \Rightarrow \partial W = M \sqcup (-S^n)$.

$W' = W \cup_{S^n} D^{n+1}$

$\Rightarrow \partial W' = M$

Since W def. retracts to S^n,
W' def retracts to D^{n+1} $\rightsquigarrow$ $\{*\}$.

$\Leftarrow$) Suppose $M = \partial W'$ with W' contractible.



$W = W' \setminus D^{n+1}$. Then $\partial W = M \sqcup (-S^n)$

We have to check that W def. ret. to M, S^n?

Consider $(D^{n+1}, S^n) \hookrightarrow (W', W)$

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(S^n) \arrow[r] \arrow[d] & H_i(D^{n+1}) \arrow[r] \arrow[d, "\cong"] & H_i(D^{n+1},S^n) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_i(W) \arrow[r] & H_i(W') \arrow[r] & H_i(W',W) \arrow[r] & \cdots
\end{tikzcd}$$

By the 5-Lemma, $S^n \hookrightarrow W$ induces isos. in homology.

$\overset{HWT}{\Longrightarrow}$ $S^n \hookrightarrow W$ is a weak htpy equivalence

$\Rightarrow$ $S^n \hookrightarrow W$ is a htpy equivalence

$\Rightarrow$ W def. ret. to S^n.

By PD: $H^k(W,S^n) \xrightarrow{\cong} H_{n+1-k}(W,M)$

$\Rightarrow$ $H_*(W,M)$ $\Rightarrow$ $M \hookrightarrow W$ induces isos in homology

$\Rightarrow$ .. $\Rightarrow$ W def. ret. to M. $\square$

2. S-parallelizability

Let's make a parenthesis.

Def: M mfld is S-parallelizable if $\tau_M \oplus \varepsilon^1$ is trivial.

S^n is known to be S-parallelizable.

Thm 2.1: A homotopy n-sphere Σ is S-parallelizable.

pf) $H^*(\Sigma; A)$ vanishes except in degrees $0, n$.

$\Rightarrow$ The only obstruction to triviality is a class

$$\omega_n(\Sigma) \in H^n(\Sigma; \pi_{n-1}(SO_{n+1})) = \pi_{n-1}(SO_{n+1}) \cong \pi_{n-1}(SO)$$

The fibration $SO_n \hookrightarrow SO_{n+1} \to S^n$ gives

$$\pi_i(SO_n) \xrightarrow{\cong} \pi_i(SO_{n+1}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} \pi_i(SO) \text{ for } n \geq i+2$$

Bott periodicity thm:

n mod 8	0	1	2	3	4	5	6	7
$\pi_{n-1}(SO)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

(i) $n \equiv 3, 5, 6, 7 \pmod 8$: $\omega_n(\Sigma) = 0$ ✓

(ii) $n \equiv 0, 4 \pmod 8$: $n = 4k$

[Kervaire] $\quad p_k(\tau_\Sigma \oplus \varepsilon^1_\Sigma) = (2k-1)!\, a_k\, \omega_n(\Sigma)$

$$\overset{\shortparallel}{p_k(\Sigma)}$$

Hirzebruch signature thm: M^{4k} closed. Then

$$\sigma(M) = \langle L_k(p_1, \dots, p_k), \mu_M \rangle$$

where $L_k(x_1, \dots, x_k) = s_k x_k +$ terms not involving x_k.

But $0 = \sigma(M) = s_k p_k = N \omega_n(\Sigma) \Rightarrow \omega_n(\Sigma) = 0$.

(iii) $n \equiv 1, 2 \pmod 8$: involves J-homomorphism. $\square$

Why do we care about S-parallelizability?

Lemma 2.2: $\xi : X \longrightarrow BSO(k)$, $k > n$. If $\xi \oplus \varepsilon^n \cong \varepsilon^{k+n}$,

n-dim CW cx
path-connected

then $\xi \cong \varepsilon^k$.

pf) Wlog $r=1$. $\xi \oplus \varepsilon^1 \cong \varepsilon^{k+1}$ gives a nullhomotopic map $i \circ \xi : X \longrightarrow BSO_{k+1}$

pointed : choose a 0-cell x_0 and make the homotopy preserve basepoints by the HEP.

Consider the fibration

$$S^k = SO_{k+1}/SO_k \xrightarrow{j} BSO_k \xrightarrow{i} BSO_{k+1}$$

$$\rightsquigarrow \quad [X, S^k]_* \xrightarrow{j_*} [X, BSO_k]_* \xrightarrow{i_*} [X, BSO_{k+1}]_*$$

$$[f] \longmapsto [\xi] \longmapsto [i \circ \xi] = [c]$$

$f : X \to S^k \xrightarrow[\text{approx.}]{\text{cellular}} f \simeq c \Rightarrow \xi \simeq c \Rightarrow \xi$ trivial.

$\square$

Cor 2.3: $M^n \subset S^{n+k}$ submfld, $k > n$.

M is S-parallelizable $\iff$ ν is trivial.

↑ normal bundle

In particular, normal bundles of homotopy n-spheres $\Sigma^n \hookrightarrow S^{n+k}$ are trivial.

Pf) $\tau_M \oplus \nu \cong \tau_{S^{n+k}}|M \cong \varepsilon^{n+k}$

$\Rightarrow$) $\tau_M \oplus \varepsilon^1 \cong \varepsilon^{n+1} \Rightarrow \underbrace{\tau_M \oplus \nu \oplus \varepsilon^1}_{\varepsilon^{n+k+1}} \cong \varepsilon^{n+1} \oplus \nu$

$\Rightarrow$ ν is trivial.

$\Leftarrow$) $\tau_M \oplus \varepsilon^{k+1} \cong \tau_M \oplus \varepsilon^1 \oplus \nu \cong \varepsilon^{n+k+1}$

$\Rightarrow$ $\tau_M \oplus \varepsilon^1$ is trivial. $\square$

Cor 2.4: M^n connected, $\partial M \neq \emptyset$

M is S-parallelizable $\iff$ M is parallelizable.

Pf) $\Leftarrow$) Clear

$\Rightarrow$) WTS: $\tau_M \cong \varepsilon^n$

As in Lemma 2.2, $f: M \to S^n$. WTS: $f \simeq c$.

$$[M, S^n] \cong [M, K(\mathbb{Z}, n)^{(n+1)}] \cong [M, K(\mathbb{Z}, n)] \cong$$

$$\cong H^n(M; \mathbb{Z}) = 0. \quad \square$$

3. bP_{n+1} & Θ_n / bP_{n+1}

Def: A homotopy n-sphere represents an element of bP_{n+1} if it is the boundary of a parallelizable mfld.

Goal: $bP_{n+1} \subset \Theta_n$ is a subgroup.

Idea: Define a group homom. $\Theta_n \xrightarrow{p} \pi_n(S) / p(S^n)$

so that $bP_{n+1} = \ker p$.

Let M^n be an S-parallelizable mfld. Pick an embedding $i : M^n \longrightarrow S^{n+k}$, $k > n+1$.

(i is unique up to isotopy)

By Cor 2.3, its normal bundle ν is trivial.

Pontrjagin-Thom construction

(i) Pick a framing φ of ν, i.e.

$$\begin{tikzcd} E(\nu) \arrow[rr, "\varphi"] \arrow[dr] & & M \times \mathbb{R}^k \arrow[dl] \\ & M & \end{tikzcd}$$

(ii) Pick a normal nbhd $N \subset S^{n+k}$

(iii) $p(M, \varphi) : S^{n+k} \longrightarrow S^k = \mathbb{R}^k \cup \{\infty\}$

$$S^{n+k} \setminus N \longmapsto \infty$$

$$N \xrightarrow{\cong} E(\nu) \xrightarrow{\varphi} M \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

$p(M,\mathcal{U})$ defines a class in $\pi_n(S) = \pi_{n+k}(S^k)$

By varying $\mathcal{U}$, $p(M) = \{p(M,\mathcal{U})\} \subset \pi_n(S)$.

<u>Lemma 3.1</u>:

(i) $0 \in p(M) \iff M$ bounds a parallelizable mfld.

(ii) $M_1 \sim_h M_2 \Rightarrow p(M_1) = p(M_2)$

(iii) $p(M_1) + p(M_2) \subset p(M_1 \# M_2) \subset \pi_n(S)$

$$\Rightarrow \begin{cases} p(S^n) \subset \pi_n(S) \text{ is a subgroup.} \\ p(\underset{\underset{\text{htpy sphere}}{\wr}}{\Sigma}) \subset \pi_n(S) \text{ is a coset of } p(S^n) \end{cases}$$

⇝ This defines the homom.

$$\begin{tikzcd} \Theta_n \arrow[r] & \pi_n(S)/p(S^n) \\ \Sigma \arrow[r, mapsto] & p(\Sigma) \end{tikzcd}$$

<u>Cor 3.2</u>: Θ_n / bP_{n+1} is finite.

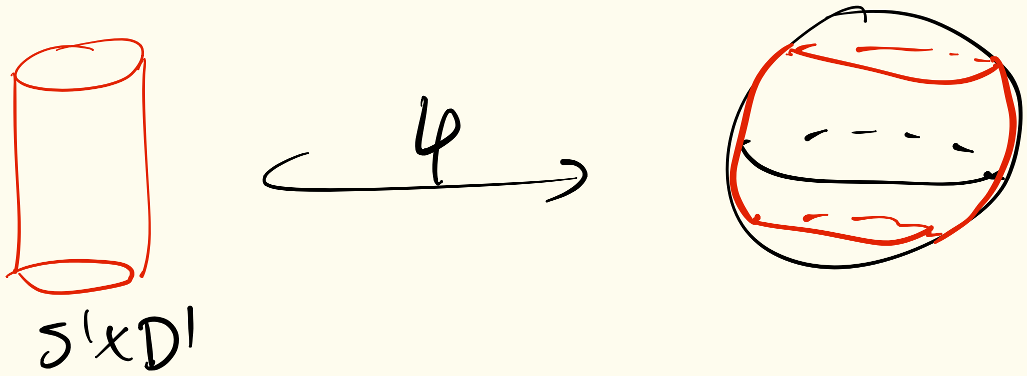
4. Computation of bP_{2k+1}

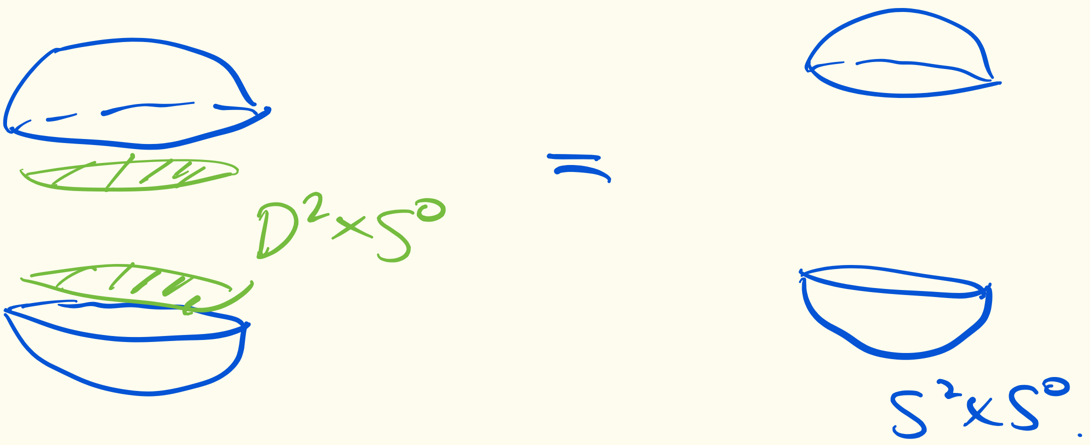
Def: Let W^n be a mfld, $\varphi: S^k \times D^{n-k} \to \mathrm{Int}\, W^n$ an embedding. $W' = \chi(W, \varphi)$ is

$$\left(W \setminus \varphi(S^k \times \{0\})\right) \underset{\substack{\varphi(u,tv) \sim (tu,v) \\ (u,v) \in S^k \times S^{n-k-1} \\ 0 < t \leq 1}}{\cup} D^{k+1} \times S^{n-k-1}$$

We say W' is obtained from W by surgery $\chi(\varphi)$.

Ex: $n=2$, $k=1$





Note: $\partial W' = \partial W$.

Suppose $\Sigma^n \in bP_{n+1}$. Then $\Sigma = \partial W$.

Goal: use surgeries to obtain W_1 highly connected with $\Sigma = \partial W_1$.

Let $\lambda \in \pi_k W$ be the class represented by $\varphi|_{S^k \times 0}$, $2k+1 < n$.

Lemma 4.1: $\pi_i W' \cong \pi_i W$ for $i<k$

$$\pi_k W' \cong \pi_k W \big/ \Lambda, \quad \lambda \in \Lambda$$

Pf) Consider $X = W \displaystyle\bigcup_{\substack{\varphi(u,y)\sim(u,y) \\ (u,y)\in S^k\times D^{n-k}}} D^{k+1}\times D^{n-k}$

X deformation retracts to $W \displaystyle\bigcup_{\varphi(u,0)\sim(u,0)} D^{k+1}\times\{0\}$

$\Rightarrow W^{(k)} = X^{(k)}$

Hence, $\pi_i W \longrightarrow \pi_i X$ iso. for $i<k$

For $i=k$: $\pi_k W \longrightarrow \pi_k X$ is $\underbrace{\text{onto}}_{\text{by CW approx}}$ and kills λ.

Similarly, $\pi_i W' \longrightarrow \pi_i X$ iso. for $i<n-k-1$

$k < n-k-1 \Rightarrow \pi_i W' \longrightarrow \pi_i X$ iso for $i<k$. □

Lemma 4.2: W^n S-parallelizable, $2k<n$. Any class $\lambda \in \pi_k W$ can be represented by some embedding

$$\varphi : S^k \times D^{n-k} \longrightarrow W$$

and hence can be killed by surgery.

Pf) [Whitney] Any $\lambda \in \pi_k W$, $2k<n$ can be represented by an embedding $\psi : S^k \longrightarrow W$.

We have $\tau_{S^k}^k \oplus \nu^{n-k} \cong \tau_W^n|S^k$

$$\Rightarrow \underbrace{\tau_{S^k}^k \oplus \varepsilon^1}_{\varepsilon^{k+1}} \oplus \nu^{n-k} \cong \tau_W^n|S^k \oplus \varepsilon^1 \overset{\substack{W \text{ S-parallelizable} \\ \downarrow}}{\cong} \varepsilon^{n+1}$$

$\overset{2.2}{\Rightarrow}$ ν^{n-k} is trivial

$\Rightarrow$ Extend ψ to a normal nbhd.

$$\rightsquigarrow \varphi : S^k \times D^{n-k} \longrightarrow W. \qquad \square$$

Lemma 4.3: φ can be chosen so that if W is S-parallelizable, so is W'.

Cor 4.4: W^n connected S-parallelizable mfld, $2k \leq n$. By a seq. of surgeries, we can obtain an S-parallelizable $(k-1)$-connected mfld W_1.

Set $n = 2k+1$, $\partial W^n = \Sigma^{2k}$ a htpy $2k$-sphere.

$$\underbrace{\pi_1, \pi_2, \dots, \pi_{k-1}}_{\text{have been killed}}, \pi_k, \pi_{k+1}, \dots, \pi_{2k+1}$$

If we kill π_k, by PD $H_* W^n = 0$

$\Rightarrow W$ is contractible and $\partial W = \Sigma^{2k}$.

$\Rightarrow bP_{2k+1} = 0$.

Let's try to kill $\pi_k W \cong H_k W$.

<u>Lemma 4.5</u>: $W' = \chi(W, \varphi)$, $\varphi : S^k \times D^{k+1} \to W$ embedding. $W_0 := W \setminus \mathrm{int}(\varphi(S^k \times D^{k+1}))$

$$\begin{tikzcd}
 & & H_{k+1} W' \arrow[d, "\cdot\lambda'"] & \varphi' : D^{k+1} \times S^k \to W' \arrow[dl] & \\
 & & H_{k+1}(W', W_0) \cong \mathbb{Z} \arrow[d, "\partial"] \arrow[dr, "\lambda"] & & \\
H_{k+1} W \arrow[r, "-\lambda"] & \mathbb{Z} \arrow[r, "\partial'"] \arrow[d, phantom, "\cong"] \arrow[dr, "\lambda'"] & H_k W_0 \arrow[r, "i"] \arrow[d, "i'"] & H_k W \arrow[r] & 0 \\
 & H_{k+1}(W, W_0) & H_k W' \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

In particular, $H_k W \cong H_k W_0 / \partial'(\mathbb{Z})$ and

$$H_k W / \lambda(\mathbb{Z}) \cong H_k W_0 / \partial(\mathbb{Z}) + \partial'(\mathbb{Z}) \cong H_k W' / \lambda'(\mathbb{Z})$$

Claim: The free part of $H_k W$ can be killed without affecting its torsion.

pf) Suppose λ generates a $\mathbb{Z}$ summand of $H_k M$.

PD $\Rightarrow$ $\mu_1 \cdot \lambda = 1$ for some $\mu_1 \in H_{k+1}(W, \partial W)$
$= H_{k+1}(W)$.

$\Rightarrow$ $i : H_k W_0 \to H_k M$ is an isom.

$\lambda' = 0$

$\Rightarrow$ $H_k W' \cong H_k W / \lambda(\mathbb{Z})$ $\square$

Fact: For k even, surgery by $\chi(\varphi)$ necessarily changes the kth Betti number.

Suppose $H_k W$ has no free part. Let $\lambda \in H_k M$ nontrivial represented by $\varphi : S^k \times D^{k+1} \to W$

We have

$$\underbrace{H_k W / \lambda(\mathbb{Z})}_{\text{finite}} \cong \underbrace{H_k W' / \lambda'(\mathbb{Z})}_{\text{infinite}}$$

$$\rightsquigarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\lambda'} H_k W' \longrightarrow H_k W' / \lambda'(\mathbb{Z}) \longrightarrow 0$$

$$\text{torsion} \hookrightarrow \text{torsion}$$

$\Rightarrow H_k W'$ has less torsion than $H_k W$.

By the claim, the recently added free part of $H_k W'$ can be killed.

This proves:

<u>Cor. 4.6</u>: $bP_{2k+1} = 0$ for k even.